# $\Omega \cong \diamond\mathcal{W}$

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## 1 Definition

### 1.1 $\Omega$

There is a correct process $c$ such that **eventually** $\Omega_p = c$ **forever** for every correct process $p$.

### 1.2 $\diamond\mathcal{W}$

Eventual Weak Accuracy + Weak Completeness :

**Eventual weak accuracy:** *There is a time after which some* correct process is never suspected by any correct process.

**Weak Completeness:** Every faulty process is eventually permanently suspected by *some* correct process.

## 2 System Assumptions

1. Complete network topology.
2. Asynchronous identified processes: a process and its identifier are used equivalently.
3. Asynchronous reliable links (not necessarily FIFO).

## 3 Notations

- $V$: the set of processes
- $Correct \subseteq V$: the set of correct processes.
- $Faulty \subseteq V$: the set of faulty processes.
- $X_p$: the value of variable $X$ of process $p$.
- $X_p^t$: the value of variable $X$ of process $p$ at time $t$.

## 4 $T_{\Omega\to\diamond\mathcal{W}}$

**Algorithm 1** $T_{\Omega\to\diamond\mathcal{W}}$, code for every process $p$

**Function** $T_{\Omega\to\diamond\mathcal{W}}(p)$
    **return** $V \setminus \{\Omega_p\}$
**End Function**

**Question 1.** Prove that $T_{\Omega\to\diamond\mathcal{W}}$ satisfies weak completeness.

Actually, $T_{\Omega\to\diamond\mathcal{W}}$ satisfies strong completeness (as shown below). Now, strong completeness implies weak completeness.

By definition, $V = \mathit{Faulty} \,\dot\cup\, \mathit{Correct}$.

Let $p$ be any correct process. By definition of $\Omega$, there is a time after which $\Omega_p \in \mathit{Correct}$ forever. So, there is a time after $\{\Omega_p\} \subseteq \mathit{Correct}$. Hence, there is a time after which $\mathit{Faulty} \subseteq V \setminus \{\Omega_p\}$, *i.e.*, there is a time after which $\mathit{Faulty} \subseteq T_{\Omega\to\diamond\mathcal{W}}(p)$. □

**Question 2.** Prove that $T_{\Omega\to\diamond\mathcal{W}}$ satisfies eventual weak accuracy.

By definition of $\Omega$, there is a time after which $\Omega_p = c$ for every correct process $p$ where $c \in \mathit{Correct}$.

Hence, by definition of the algorithm, there is a time after which $c \notin T_{\Omega\to\diamond\mathcal{W}}(p)$ for every correct process $p$. In other words, there is a time after which the correct process $c$ is never more suspected by every correct process $p$. □

# 5 $T_{\diamond\mathcal{W}\to\Omega}$

**Question 1.** Following the principles presented in the lesson, propose an algorithm $T_{\diamond\mathcal{W}\to\Omega}$.

*Notations:* Please use the following variables:

- $\mathit{Leader} \in V$, initialized to $p$.
- $C[]$, array of integers indexed on $V$, every cell is initialized to 0.

**Algorithm 2** $T_{\diamond\mathcal{W}\to\Omega}$, code for every process $p$

**Variables:**
    $\mathit{Leader} \in V$, initialized to $p$
    $C[]$, array of integers indexed on $V$, every cell is initialized to 0
**End Variables**

**Function** $T_{\diamond\mathcal{W}\to\Omega}(p)$
    **return** $\mathit{Leader}$
**End Function**

**while** *true* **do**     /* *Must be run into a separated thread* */
    `broadcast` $\langle C, p\rangle$ `to` $V \setminus \{p\}$
    **For all** $q \in V \setminus \{p\}$ **do**
        **If** `receive` $\langle CN, q\rangle$ **then**
            **For all** $x \in V$ **do**
                $C[x] \leftarrow \max(C[x], CN[x])$
            **End For**
        **End If**
    **End For**
    **For all** $q \in \diamond\mathcal{W}(p)$ **do**
        $C[q]++$
    **End For**
    $\mathit{Leader} \leftarrow \min\{q \in V \mid \forall q' \in V, C[q] \leq C[q']\}$
**end while**

**Question 2.** Show the following lemma.

**Lemma 1.** *$\forall p \in Correct$, $\forall q \in Faulty$, $\forall t \in \mathbb{N}$, $\exists t' > t$ such that $C[q]_p^t < C[q]_p^{t'}$.*

*Proof.* First, by definition of the algorithm, $C[q]_p$ is monotonically nondecreasing with time, *i.e.*, $C[q]_p^t \leq C[q]_p^{t'}$, $\forall t, \forall t' > t$.

Then, since $\diamond\mathcal{W}$ satisfies weak completeness, there is a correct process $r$ and a time $t_0$ such that $\forall t \geq t_0$, $q \notin \diamond\mathcal{W}(r)^t$.

Since $r$ is correct, $r$ executes $C[q]++$ infinitely often, *i.e.*, $C[q]_r$ regularly increases. Moreover, $r$ broadcasts $C_r$ infinitely often and, as the links a reliable, every correct process $p \neq r$ receives $C_r$ infinitely often. By Line 13, $C[q]_p$ also regularly increases and we are done. □

**Question 3.** Show the following lemma.

**Lemma 2.** *$\exists p \in Correct$ such that $\exists k, t \in \mathbb{N}$ such that $\forall q \in Correct$, $\forall t' \geq t$, $C[p]_q^{t'} \leq k$.*

*Proof.* Since $\diamond\mathcal{W}$ satisfies eventual weak accuracy, $\exists \ell \in Correct$, $\exists t_\alpha \in \mathbb{N}$ such that $\forall p \in Correct$, $\forall t' \geq t_\alpha$, $\ell \notin \diamond\mathcal{W}(p)$.

Let $t_\beta$ be the time from which every faulty process has crashed and every message sent by a faulty process to a correct process has been received. Let $t_0 = \max(t_\alpha, t_\beta)$. Let $Max = \max_{p \in Correct} C_p^{t_0}[\ell]$.

We now prove by induction on $t$ that $\forall t \geq t_0$, $\forall p \in Correct$, $C_p^t[\ell] \leq Max$ and every message $\langle CN, \_\rangle$ in transit to $p$ at time $t$ satisfies $CN[\ell] \leq Max$.

**Base Case:** $t = t_0$. By Definition, $\forall p \in Correct$, $C_p^{t_0}[\ell] \leq Max$. Moreover, since $C_p[\ell]$ is monotonically nondecreasing with time and no message in transit at time $t_0$ has been sent by a faulty process (by definition), every message $\langle CN, \_\rangle$ in transit to $p$ at time $t_0$ satisfies $CN[\ell] \leq Max$.

**Induction Step:** Consider the step from time $t \geq t_0$ to time $t + 1$. Let $p \in Correct$.

If $p$ does not receive any message in the step, then $C_p[\ell]$ stays unchanged since by definition $p$ never more execute Line 18. So, $C_p^{t+1}[\ell] \leq MAX$ by induction hypothesis.

Otherwise, $C_p[\ell]$ is assigned to the maximum value between $C_p[\ell]^t$ and the values $CN[\ell]$ of every received message at time $t$. By induction hypothesis, all these values are less than or equal to $Max$. Hence, $C_p^{t+1}[\ell] \leq MAX$.

Finally, since every message in transit at time $t+1$ has been sent before $t+1$, from the induction hypothesis, the previous claim and owing the fact that only correct processes can send messages during the step from time $t$ to time $t + 1$, we can deduce that every message $\langle CN, \_\rangle$ in transit to any correct process at time $t + 1$ satisfies $CN[\ell] \leq Max$.

Thus, the induction holds at time $t + 1$ and we are done.

Hence, the lemma holds with $p = \ell$. □

**Question 4.** Show the following lemma.

**Lemma 3.** *$\exists t \in \mathbb{N}$ such that $\forall p \in Correct$, $\forall t' \geq t$, $C_p^t[Leader_p^t] = C_p^{t'}[Leader_p^{t'}]$.*

*Proof.* Assume, by the contradiction, that $\forall t \in \mathbb{N}$, $\exists p \in Correct$ and $\exists t' \geq t$ such that $C_p^t[Leader_p^t] \neq C_p^{t'}[Leader_p^{t'}]$.

First, since the number of correct process is finite, $\exists p \in Correct$ such that $\forall t \in \mathbb{N}$, $\exists t' \geq t$ such that $C_p^t[Leader_p^t] \neq C_p^{t'}[Leader_p^{t'}]$.

Moreover, since $C_p[q]$ is monotonically nondecreasing for all $q \in V$, we have: $\exists p \in Correct$ such that $\forall t \in \mathbb{N}$, $\exists t' \geq t$ such that $C_p^t[Leader_p^t] < C_p^{t'}[Leader_p^{t'}]$.

By Line 20 and owing the fact that $V$ is finite, we have $\forall q \in V$, $\forall t \in \mathbb{N}$, $\exists t' \geq t$ such that $C_p^t[q] < C_p^{t'}[q]$. Now, since $p$ is correct, $p$ sends its counter array infinitely often to all processes. Moreover, since links are reliable, all other correct processes receive infinitely many messages containing counter arrays from $p$. Hence, by Line 13, $\forall r \in Correct$, $\forall q \in V$, $\forall t \in \mathbb{N}$, $\exists t' \geq t$ such that $C_r^t[q] < C_r^{t'}[q]$, which contradicts Lemma 2. □

**Question 5.** Show the following corollary.

**Corollary 1.** *$\exists t \in \mathbb{N}$ such that $\forall p \in Correct$, $\forall t' \geq t$, $Leader_p^t = Leader_p^{t'}$.*

*Proof.* Assume, by the contradiction that $\forall t \in \mathbb{N}$, $\exists p \in Correct$, $\exists t' \geq t$ such that $Leader_p^t \neq Leader_p^{t'}$.

Since the number of correct processes is finite, we have:

**Claim 1:** $\exists \ell \in Correct$ such that $\forall t \in \mathbb{N}$, $\exists t' \geq t$ such that $Leader_\ell^t \neq Leader_\ell^{t'}$.

Let $t_0 \in \mathbb{N}$ such that $\forall p \in Correct$, $\forall t' \geq t_0$, $C_p^{t_0}[Leader_p^{t_0}] = C_p^{t'}[Leader_p^{t'}]$. By Lemma 3, $t_0$ is well-defined. Let $n = |V|$. By Claim 1, $\exists t_1, t_2, \ldots, t_n$ such that $\forall i \in [1..n]$, $t_i > t_{i-1}$ and $Leader_\ell^{t_{i-1}} \neq Leader_\ell^{t_i}$.

By Line 20, and owing the fact that counters are monotonically nondecreasing, $\forall i \in [0..n-1]$, $C_\ell^{t_{i+1}}[Leader_\ell^{t_i}] > C_\ell^{t_0}[Leader_\ell^{t_0}]$. Hence, by definition of $t_0$, $\forall i \in [0..n-1]$, $\forall j \in [i+1..n]$, $Leader_\ell^{t_i} \neq Leader_\ell^{t_j}$.

Hence, there are at least $n+1$ distinct processes, a contradiction. □

**Question 6.** Show the following lemma.

**Lemma 4.** *$\exists \ell \in Correct$ and $\exists t \in \mathbb{N}$ such that $\forall t' \geq t$, $\forall p \in Correct$, $Leader_p^{t'} = \ell$.*

*Proof.* Let $t_z \in \mathbb{N}$ such that $\forall p \in Correct$, $\forall t' \geq t_z$, $Leader_p^{t_z} = Leader_p^{t'} \wedge C_p^{t_z}[Leader_p^{t_z}] = C_p^{t'}[Leader_p^{t'}]$. By Lemma 3 and Corollary 1, $t_z$ is well-defined.

By Lemma 1, $\forall p \in Correct$, $Leader_p^{t_z} \in Correct$.

Assume now, by the contradiction, that $\exists p_1, p_2 \in Correct$ such that $Leader_{p_1}^{t_z} \neq Leader_{p_2}^{t_z}$.

Without the loss of generality, assume that $C_{p_1}^{t_z}[Leader_p^{t_z}] < C_{p_2}^{t_z}[Leader_{p_2}^{t_z}] \vee (C_{p_1}^{t_z}[Leader_{p_1}^{t_z}] = C_{p_2}^{t_z}[Leader_{p_2}^{t_z}] \wedge p_1 < p_2)$.

Then, $C_{p_2}^{t_z}[Leader_{p_1}^{t_z}] > C_{p_2}^{t_z}[Leader_{p_2}^{t_z}] \geq C_{p_1}^{t_z}[Leader_{p_1}^{t_z}]$.

Since $p_2$ is correct and the counters are monotonically nondecreasing, $p_2$ sends infinitely many $\langle CN, p_2 \rangle$ messages to $p_1$ with $CN[Leader_{p_1}^{t_z}] > C_{p_1}^{t_z}[Leader_{p_1}^{t_z}]$. Since the link are reliable and $p_1$ is correct, by Line 13, $\exists t' > t_z$ such that $C_{p_1}^{t_z}[Leader_{p_1}^{t_z}] < C_{p_1}^{t'}[Leader_{p_1}^{t_z}] = C_{p_1}^{t'}[Leader_{p_1}^{t'}]$ (by definition of $t_z$), a contradiction.

Hence, $\forall p, q \in Correct$, $Leader_p^{t_z} = Leader_q^{t_z}$, and by definition of $t_z$, the lemma holds. □